The Divergence in Coordinate Systems

Consider now the **divergence** of vector fields expressed with our **coordinate systems**:

Cartesian

$$\nabla \cdot \mathbf{A}(\bar{r}) = \frac{\partial A_x(\bar{r})}{\partial x} + \frac{\partial A_y(\bar{r})}{\partial y} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Cylindrical

$$\nabla \cdot \mathbf{A}(\bar{r}) = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho(\bar{r})\right)}{\partial \rho}\right] + \frac{1}{\rho}\frac{\partial A_\phi(\bar{r})}{\partial \phi} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Spherical

$$\nabla \cdot \mathbf{A}(\bar{r}) = \frac{1}{r^2}\left[\frac{\partial\left(r^2 A_r(\bar{r})\right)}{\partial r}\right] + \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta\, A_\theta(\bar{r})\right)}{\partial \theta}\right] + \frac{1}{r\sin\theta}\frac{\partial A_\phi(\bar{r})}{\partial \phi}$$

Note that, as with the gradient expression, the divergence expressions for cylindrical and spherical coordinate systems are more **complex** than those of Cartesian. Be **careful** when you use these expressions!

For **example**, consider the vector field:

$$\mathbf{A}(\bar{\mathrm{r}}) = \frac{\sin\theta}{r}\hat{a}_r$$

Therefore, $A_\theta = 0$ and $A_\phi = 0$, leaving:

$$\begin{aligned}
\nabla \cdot \mathbf{A}(\bar{\mathrm{r}}) &= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 A_r(\bar{r})\right)\right] \\
&= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\sin\theta}{r}\right)\right] \\
&= \frac{1}{r^2}\left[\frac{\partial\left(r\sin\theta\right)}{\partial r}\right] \\
&= \frac{1}{r^2}[\sin\theta] = \frac{\sin\theta}{r^2}
\end{aligned}$$